Special Relativity

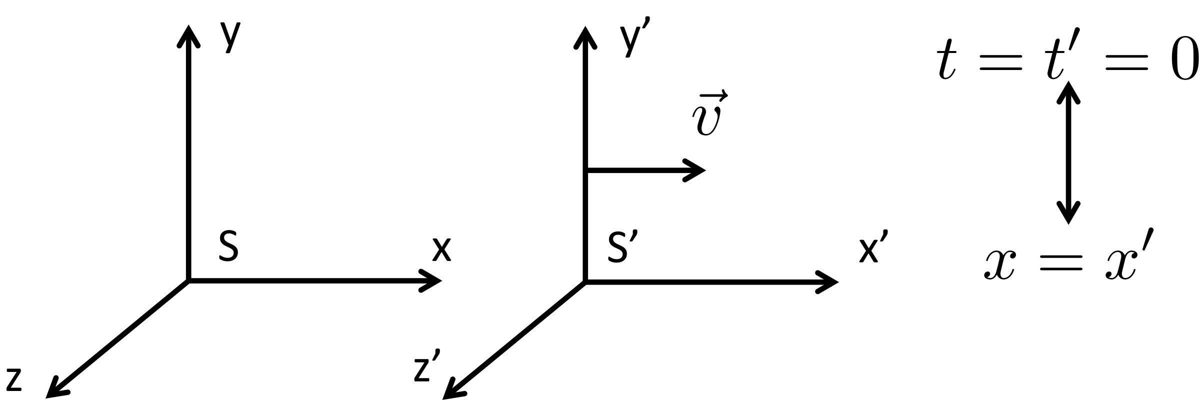
This is a summary. Not many proofs are provided
I assume you have studied Special Relativity

Here I follow D. Griffiths "Introduction to Elementary Particles"

Postulates of special relativity

1. The laws of physics apply to any inertial reference frame. An inertial reference frame is one on which Newton's 1st law applies.
2. The speed of light, c = 2.99793 x 10^8 m/s is independent of the choice of inertial frame.

Imagine 2 frames S & S', with S' moving with velocity $\vec{v}$ with respect to S. We choose the coordinate system thus:



Lorentz Transformations

Given an event at (x, y, z, t) as seen from S, the observations from S' are given using Lorentz transformations':

$$x' = \gamma(x - vt)$$

$$y' = y \quad z' = z$$

$$t' = \gamma(t - v/c^2 x)$$

where:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \beta = v/c$$

The inverse transformation is a simple change of sign

$$x = \gamma(x' + vt')$$

$$y = y' \quad z = z'$$

$$t = \gamma(t' + v/c^2 x')$$

Some consequences

1. Relativity of simultaneity. If two events happen simultaneously in S, but at different locations, they are not simultaneous in S':
$$t'_a - t'_b = \gamma \frac{v}{c^2}(x_a - x_b)$$
2. Lorentz contraction. Assume a bar at rest in S', so that one end is at x'=0 and the other end at x'=L. The length as measured in S is L/γ Only along x/x'.
3. Time dilation. Assume a time difference between events at x'=y'=z'=0 measured in S'. For simplicity t'_1=0 and t'_2=T and (or ΔT'=T). The time difference measured at S is γT

<u>Example</u>. The muon lifetime is τ_μ = 2.2 x 10^{-6} s. Muons are at produced an altitude of ~10 km, due to cosmic ray interactions with the atmosphere. Näively, the muon travels a maximum $c\tau_\mu$ = 660 m. However, a 2 GeV muon (m_μ = 106 MeV) has γ = 19, so we expect it to travel $\gamma c \tau_\mu$ =12.5 km and live $\gamma \tau_\mu$ = 4.2x10^{-5} s in Earth's reference frame. Question: explain how Lorentz contraction is observed in this example.

Some consequences

4. Velocity addition. Assume a particle moving along x' with speed u' with respect to S'. What is the speed with respect to S? It travels a distance $\Delta x = \gamma_v(\Delta x' + v\Delta_t')$ in a time

$\Delta t = \gamma_v(\Delta t' + v/c^2 \Delta x')$. So the velocity is:

$$\frac{\Delta x}{\Delta t} = \frac{\Delta x' + v\Delta t'}{\Delta t' + v/c^2 \Delta x'} = \frac{\Delta x'/\Delta t' + v}{1 + v/c^2 \Delta x'/\Delta t'}$$

or:

$$u = \frac{u' + v}{1 + vu'/c^2}$$

Note that I label, the Lorentz boost factor. Now γ_v, γ_u and $\gamma_{u'}$ are possible

Velocity does not transform using Lorentz …

This is only for the particle moving along x'.

4-vectors

It's convenient to describe space time as a 4-vector (x_0, x_1, x_2, x_3) where $ct = x_0$

A change of inertial frame can be described like this $x^{\mu\prime} = \Lambda^\mu_\nu x^\nu$

Here Λ^μ_ν is a Lorentz transformation and there's an implicit summation over repeated indices. Example boost along x:

$$\Lambda = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The Lorentz transformations form a group, SO(3,1), that leave the norm of a 4-vector invariant:

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = x_0'^2 - x_1'^2 - x_2'^2 - x_3'^2$$

This invariance follows from the constancy of c in all inertial frames

4-vectors

Indices are raised and lowered, using the metric

$$x^\nu = g_{\nu\mu} x^\mu \qquad x_\nu = g^{\nu\mu} x_\mu$$

Where the metric is

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Which allows for a compact notation for the square of the norm

$$x^\nu x_\nu = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

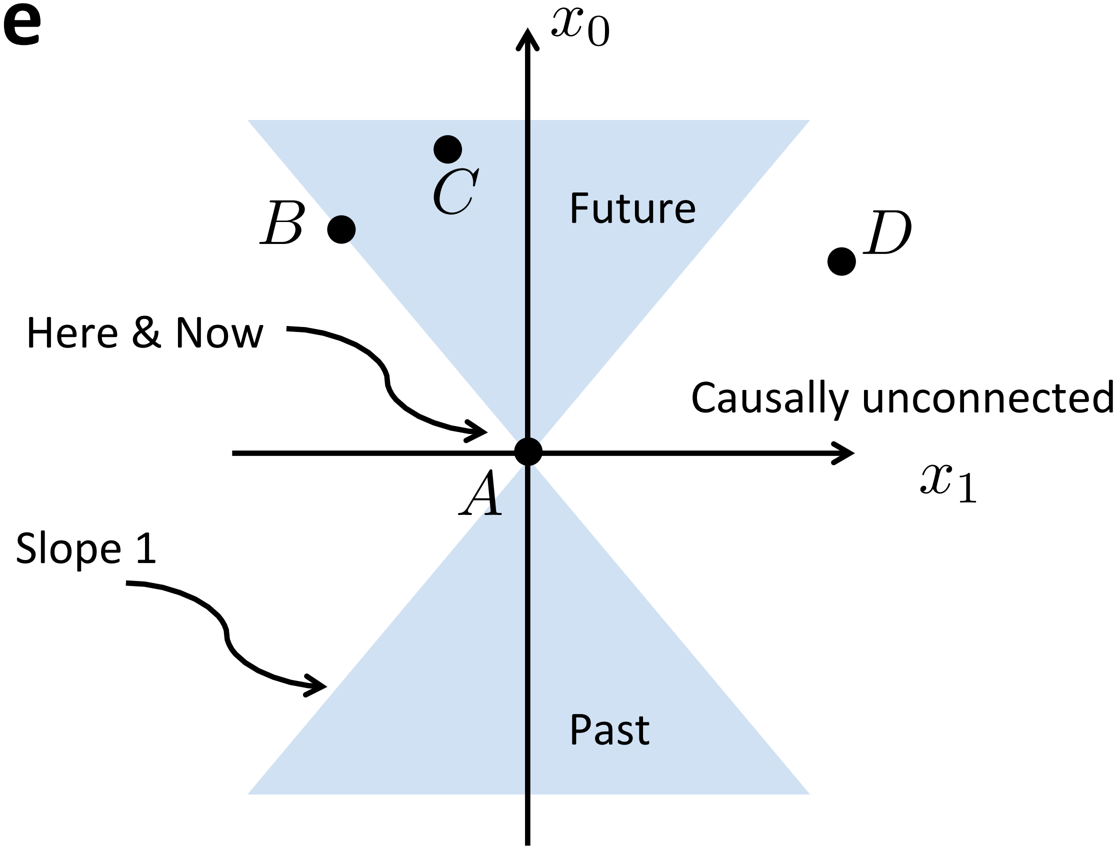
And the dot product of two 4-vectors $a^\nu b_\nu = a \cdot b$

If … $a^\nu a_\nu > 0$ a is "time-like"

$a^\nu a_\nu = 0$ a is "light-like"

$a^\nu a_\nu < 0$ a is "space-like"

Light cone



The difference between events A & B is light-light. They are causally connected by c.

The difference between events A & C is time-light. There is a reference frame in which the difference between A & C is only in time. They can be causally related.

The difference between events A & D is space-light. These events cannot be connected causally. There is a reference frame for which A & D are simultaneous.

Energy and Momentum

Velocity does not transform following a Lorentz transformation.

However $\eta = \gamma(c, v_x, v_y, v_z)$ is a 4-vector with $\eta_\mu \eta^\mu = c^2$

We can define the 4-momentum as $p = m\eta$, where:

$$p_\mu p^\mu = m^2 c^2$$

The components of the 4-momentum are interpreted as:

$$p = (\gamma mc, \gamma m v_x, \gamma m v_y, \gamma m v_z) = (E/c, \vec{p})$$

From which it follows: $mc = \sqrt{E^2/c^2 - |\vec{p}|^2}$

Note that for $\vec{p} = 0$, the rest energy is $E_{rest} = mc^2$

The kinetic energy is defined as:

$$T = E - mc^2 = (\gamma - 1)mc^2$$

For a massless particle $E = |\vec{p}|/c$, which implies that the 4-vector is light-like (very satisfyingly light is light-like).

Energy and Momentum

Example. Muons are at produced an altitude of ~10 km, due to cosmic ray interactions with the atmosphere. A 2 GeV muon (m_μ = 106 MeV) $\gamma = E/m_\mu c^2 = 19$. Typical muon energy at the ground is a few GeV.

Energy and momentum conservation can be summarized as 4-momentum conservation. Specifically, “The total 4-momentum of an isolated system is constant”.

Target and C.M. frames

A common reference frames is the target frame

$$A + B \rightarrow C + D + \ldots .$$

where B is at rest. In this case the invariant mass is:

$$Wc^2 = \sqrt{(E_A + M_B c^2)^2 - c^2|\vec{p}_A|^2} = \sqrt{2E_A M_B c^2 + M_A^2 c^4 + M_B^2 c^2}$$

Which in the limit $E_A >> M_A c^2$, $M_B c^2$ is $Wc^2 = \sqrt{2E_A M_B c^2}$

Another common frame is the Center of Mass frame with

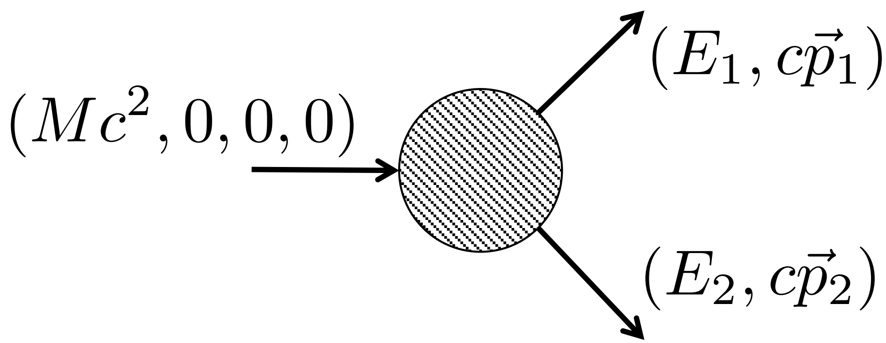
$$\vec{p}_A + \vec{p}_B = 0$$

and thus $Wc^2 = E_A + E_B$

(This is why particle physicist build colliders)

Two body decay

A particle of mass M decays into two daughter particles of masses m_1 and m_2. Note that $M \geq m_1 + m_2$. It's easiest to look at the decay in the rest frame of the mother particle.



4-momentum conservation: $Mc^2 = E_1 + E_2 \quad 0 = \vec{p}_1 + \vec{p}_2$

Norm-invariance
$$m_1^2 c^4 = E_1^2 + |\vec{p}_1|^2 c^2$$
$$m_2^2 c^4 = E_2^2 + |\vec{p}_2|^2 c^2$$

From which it follows:

$$c|\vec{p}_1| = \sqrt{E_1^2 - m_1^2 c^4} \quad c|\vec{p}_2| = \sqrt{E_2^2 - m_2^2 c^4}$$

Two body decay

$$c|\vec{p}_1| = \sqrt{E_1^2 - m_1^2 c^4} = c|\vec{p}_2| = \sqrt{(Mc^2 - E_1)^2 - m_2^2 c^4}$$

$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} c^2 \quad E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M} c^2$$

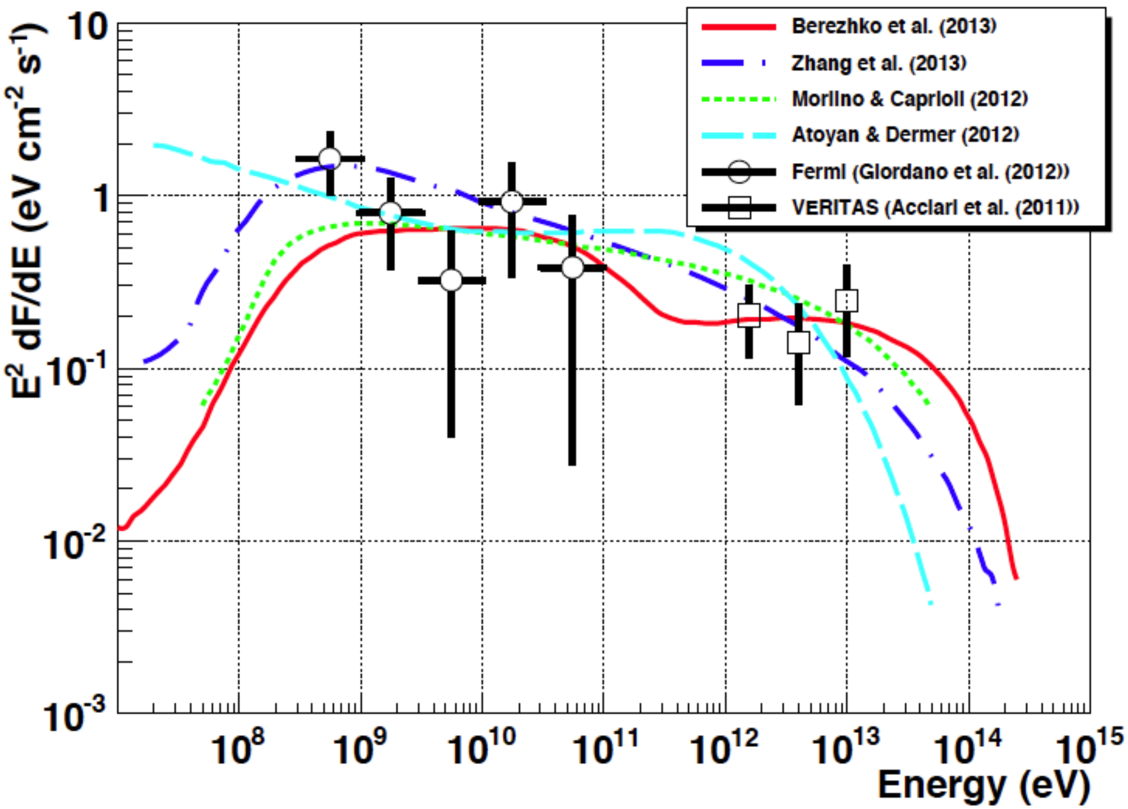
Example: The main decay mode (B=99.98%) of the charged pion is: $\pi^+ \to \mu^+ \nu_\mu$

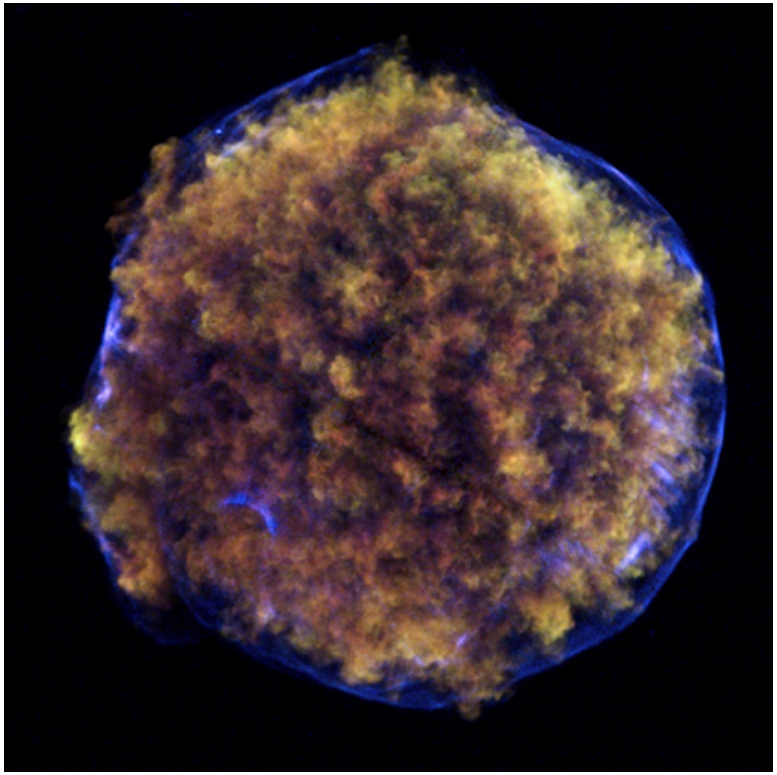
The masses are m_{π^+}= 140 MeV, m_μ = 106 MeV, m_ν ~ 0. Then, in the rest frame of the pion, the energies of the muon and neutrino are:

E_μ = 110 MeV ($\gamma \approx 1.038$) E_ν = 29.9 MeV

Example: The main decay mode (B=98.8%) of the neutral pion is: $\pi_0 \to \gamma\gamma$. The mass of the neutral pion is m_{π^0} = 136 MeV. From which it follows $E_\gamma = m_{\pi^0}/2 =$ 68 MeV.

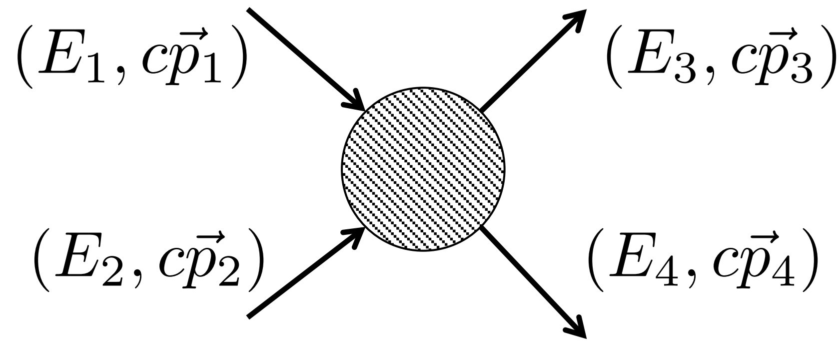
Tycho's supernova – "pion bump"





N. Park, et al. ICRC 2013 Proc.
arXiv:1308.0610

Two -> Two body collision



Let's work in the center of mass rest frame, i.e.

$$\vec{p}_1 + \vec{p}_2 = 0 = \vec{p}_3 + \vec{p}_4$$

Now:

$$(E_1, c\vec{p}_1) + (E_2, c\vec{p}_2) = (E_1 + E_2, 0, 0, 0)$$

In this frame we define the invariant mass of the system as

$$Wc^2 = E_1 + E_2$$

Note that more generally

$$Wc = \sqrt{(p_1 + p_2)^\mu (p_1 + p_2)_\mu}$$

W is a Lorentz invariant.

It's easy to generalize the concept to more than 2 particles. And note that W can be calculated in terms of particles 1 & 2 or 3 & 4

Two -> Two body collision

4-momentum conservation

$$Wc^2 = E_1 + E_2 = E_3 + E_4 \quad \vec{p}_1 + \vec{p}_2 = 0 \quad \vec{p}_3 + \vec{p}_4 = 0$$

4-momentum norm invariance for each particle:

$$m_i^2 c^4 = E_i^2 + |\vec{p}_i|^2 c^2 \quad i = 1, 2, 3, 4$$

From which it follows:

$$c|\vec{p}_1| = \sqrt{E_1^2 - m_1^2 c^4} \quad c|\vec{p}_2| = \sqrt{E_2^2 - m_2^2 c^4}$$

$$c|\vec{p}_1| = \sqrt{E_1^2 - m_1^2 c^4} = c|\vec{p}_2| = \sqrt{(Wc^2 - E_1)^2 - m_2^2 c^4}$$

(and similar for particles 3 & 4). Then:

$$E_1 = \frac{W^2 + m_1^2 - m_2^2}{2W} c^2 \quad E_2 = \frac{W^2 + m_2^2 - m_1^2}{2W} c^2$$

$$E_3 = \frac{W^2 + m_3^2 - m_4^2}{2W} c^2 \quad E_4 = \frac{W^2 + m_4^2 - m_3^2}{2W} c^2$$

In the CM frame

Hey! This looks familiar!

Extragalactic Background Light attenuation

The Universe is filled with IR photons (say 1 eV) due to starlight. For photons of energy high enough it's possible to create e^+e^- pairs thus:

$$\gamma_{HE} + \gamma_{IR} \rightarrow e^+ + e^-$$

Above which energy are high energy photons attenuated?

The minimum energy of production is such that in the C.M. frame, the e^+e^- pair is at rest. It follows that $W = 2m_e$

Now assuming head on collision of the 2 photons:

$$(E_{HE} + E_{IR})^2 - (E_{HE} - E_{IR})^2 = 4E_{HE}E_{IR} = W^2c^4$$

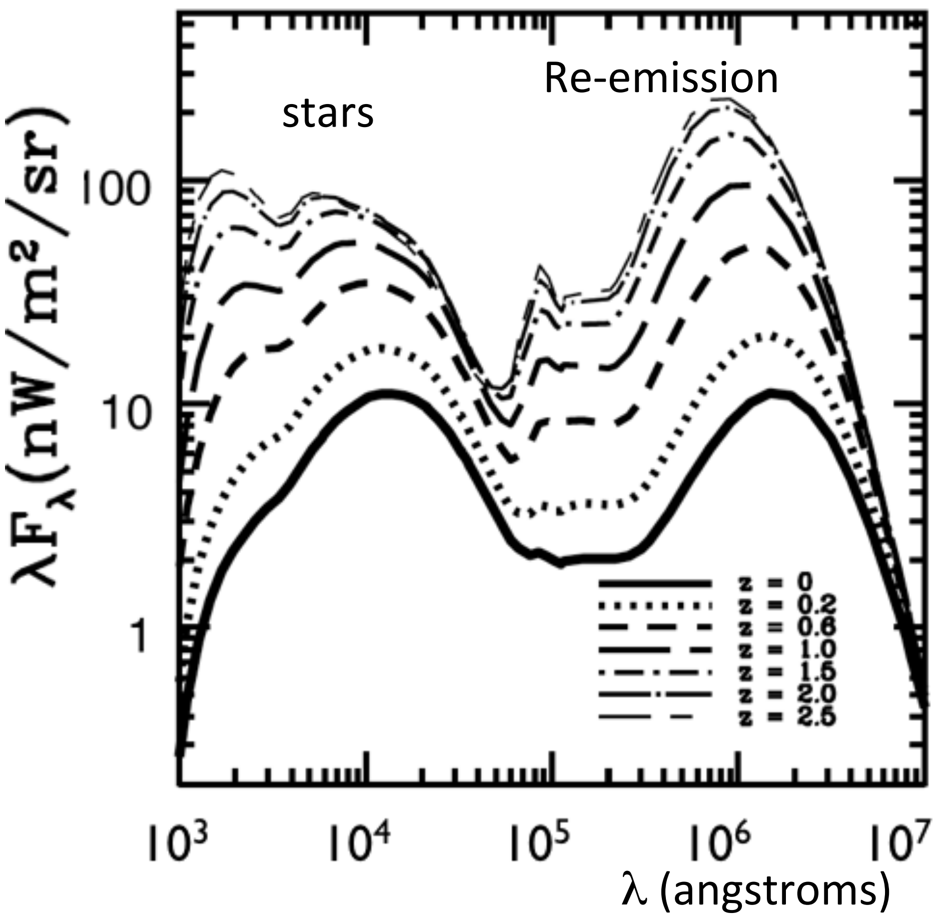
Which means that attenuation happens for:

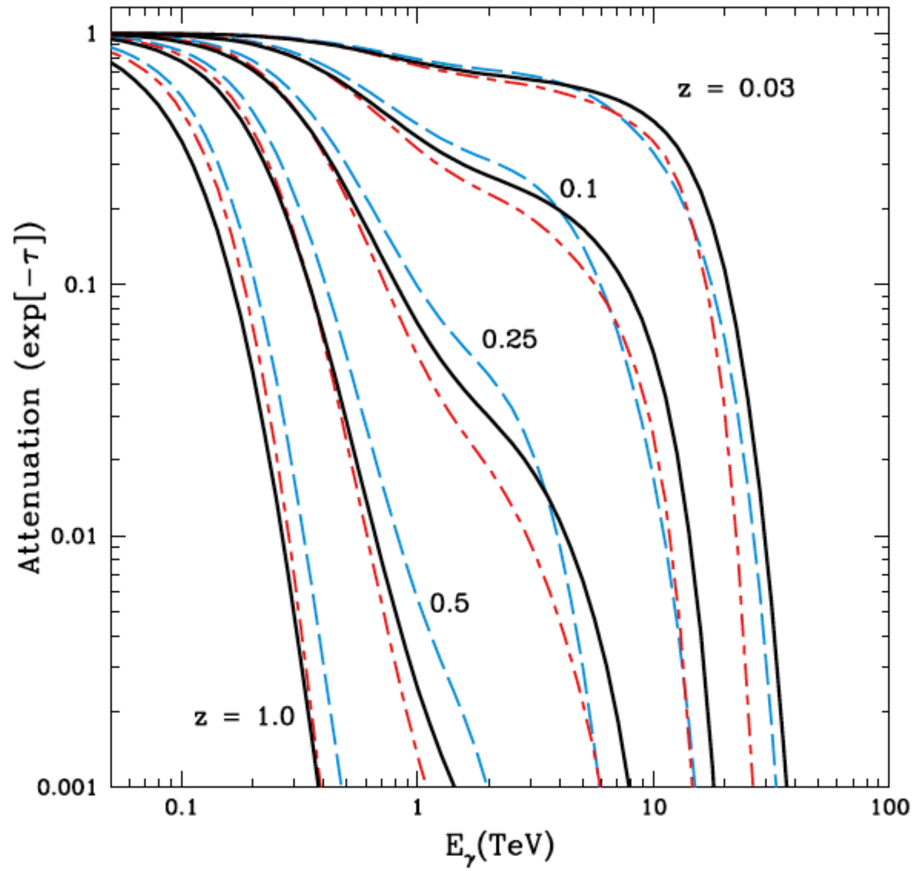
Electron mass: 511 keV

$$E_{HE} \geq \frac{W^2c^4}{4E_{IR}} = \frac{m_e^2c^4}{E_{IR}} = \text{260 GeV (}2.6 \times 10^{11}\text{ eV)}$$

Extragalactic Background Light

10^4 Angstroms = 1.24 eV. Near IR





J. Primack et al. arXiv 1107.2566

Reaction Thresholds

Assume a beam A and a target B

$$A + B \to 1 + 2 + 3 + ...$$

What is the energy threshold, so that the reaction can take place?

The invariant masses have to be the same. i.e. $W_{AB} = W_{123...}$

The minimum value for $W_{123...}$ is $m_1 + m_2 + m_3 + ...$

$$W_{AB}^{min} = \sum_i m_i$$

For B at rest (target)

$$(E_A + m_B c^2)^2 - |\vec{p}_A|^2 c^2 = (\sum_i m_i)^2 c^4$$

which implies:

$$E_A^{min} = \frac{(\sum_i M_i)^2 - m_A^2 - m_B^2}{2m_B} c^2$$

The GZK cutoff

The Universe is filled with a background of 2.6x10^{-4} eV photons (i.e. the 3 K CMB). Cosmic rays have been observed up to ~10^{20} eV (almost 10 J!)

The following reaction

$$p + \gamma_{CMB} \rightarrow \Delta^+ \rightarrow \begin{array}{c} n + \pi^+ \\ p + \pi^0 \end{array}$$

is possible when the invariant mass of the proton and CMB exceeds the pion production threshold (not the same as the delta resonance). At the threshold (head on), the invariant mass is:

$$(m_{\pi_0} + m_p)^2 c^4 = W^2 c^4 = |(E_p, cp, 0, 0) + (E_\gamma, -E_\gamma, 0, 0)|^2$$

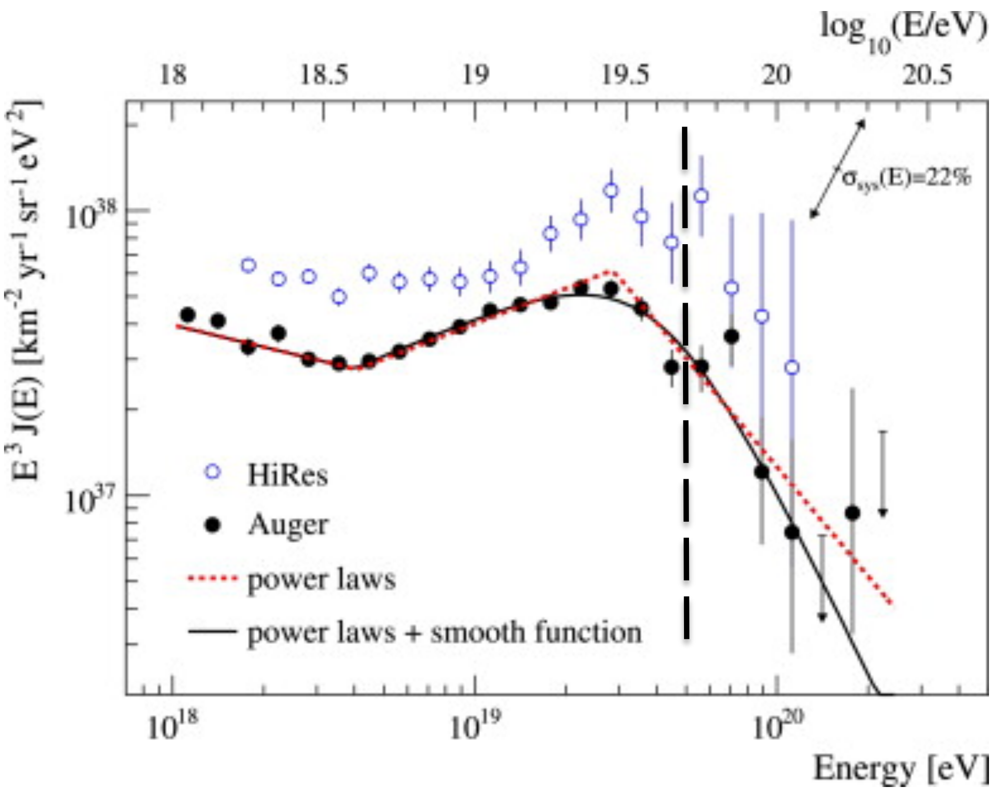
Developing this and using: $E_p^2 - c^2 p^2 = m_p^2 c^4$

$$(m_{\pi_0} + m_p)^2 c^4 - m_p^2 c^4 = 2E_p E_\gamma + 2cpE_\gamma$$

The GZK cutoff

Approximating $cp \approx E_p$ $$E_p = \frac{m_{\pi^0} + 2m_{\pi^0}m_p}{4E_\gamma}c^4$$

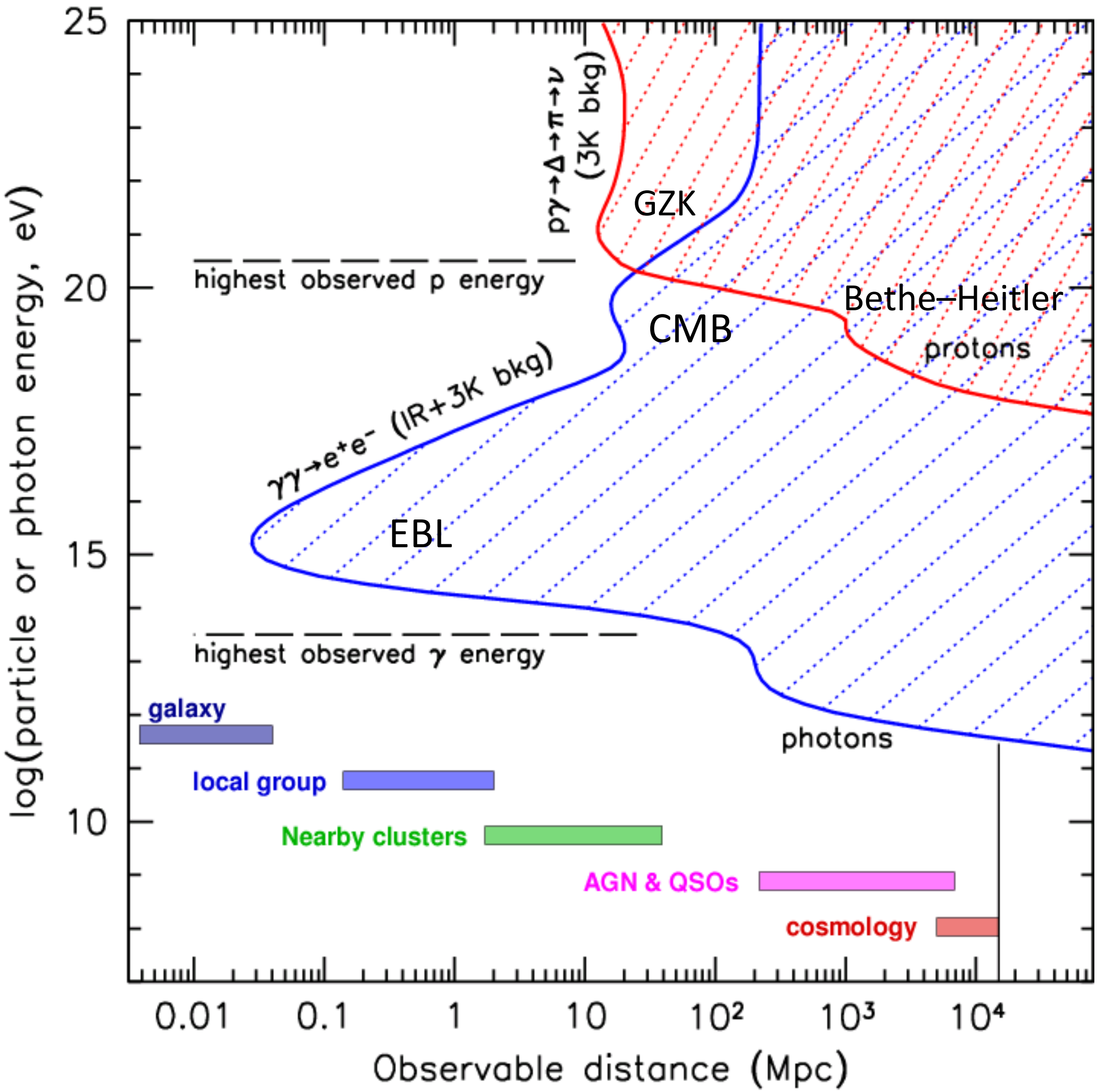
This yields 3.6 x 10^{20} eV. It's customary to use the tail of the CMB distribution. 5x higher energy photons yields a threshold of 5 x 10^{19} eV. (If you use peak CMB energy and delta resonance you get ~6 x 10^{20} eV - math check needed.)



Pierre Auger Collaboration
Phys Lett B, 685 (2010) 239-246

The observed cutoff may or may not be due to GZK. More later ...

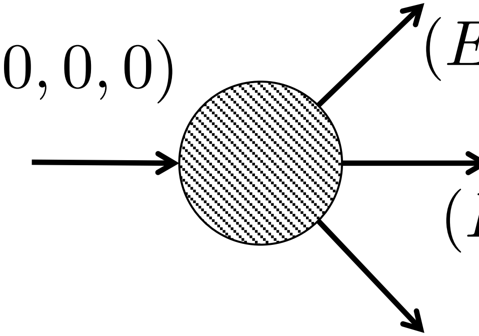
Universe opacity to high energy particles



$$p + \gamma \to p + e^+ + e^-$$

Whether Bethe-Heitler has been seen in the C.R. spectrum is contentious. More later …

Three body decay



Clearly for 3-body decays, the output 4-momenta are not uniquely set.

We can instead study the interesting ranges

Trivially, in the mother's rest frame, the minimum energy for particle i is $m_i c^2$. And the other two particles follow a 2-body decay from a mother of $(M\text{-}m_i)$ mass.

Maximum energy? Note that in the two body decay the energy of particle 1 is the largest

$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} c^2$$

when m_2 is smallest.

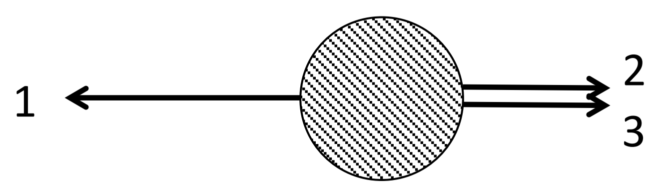
Three body decay

We now represent the three body decay as $M \to 1 + (23)$ with (23) having a "mass" (invariant mass actually) of W_{23}.

Then the energy of 1 is largest when W_{23} is smallest. W_{23} is smallest for 2 & 3 having the same velocity vector – no relative kinetic energy – and $W_{23} = m_2 + m_3$

$$E_1^{max} = \frac{M^2 + m_1^2 - (m_1 + m_2)^2}{2M} c^2$$

And similar for 2 & 3



Proof

In general: $W_{23}^2 c^4 = (E_2 + E_3)^2 - c^2|\vec{p}_2 + \vec{p}_3|^2$

Also: $E = \gamma m c^2 \qquad \vec{p} = \gamma m \vec{v}$

If both particles have the same velocity:

$$W_{23}^2 c^4 = (\gamma m_2 c^2 + \gamma m_3 c^2)^2 - c^2|\gamma m_2 \vec{v} + \gamma m_3 \vec{v}|^2$$

$$W_{23}^2 c^4 = \gamma^2 (m_2 + m_3)^2 c^4 - \gamma^2 (m_2 + m_3)^2 c^2 |\vec{v}|^2$$

$$W_{23}^2 c^4 = \gamma^2 (m_2 + m_3)^2 c^2 (c^2 - |\vec{v}|^2)$$

$$W_{23}^2 c^4 = (m_2 + m_3)^2 c^4$$

(Q.E.D.)